October Math Gems

PROBLEM OF THE WEEK 15

§1 Problems

Problem 1.1. Show that in any triangle ABC we have $\frac{a\cos C - b\cos B}{a\cos B - b\cos A} + \cos C = 0$

Solution. From sin law, $a = 2R\sin A, b = 2R\sin B$

$\frac{2R\sin A\cos A - 2R\sin B\cos B}{2R\sin A\cos B - 2R\sin A\cos B} + \cos C = \frac{\sin A\cos A - \sin B\cos B}{\sin A\cos B - \sin A\cos B} + \cos C =$

$\frac{\frac{1}{2}\sin 2A - \frac{1}{2}\sin 2B}{\sin(A-B)} + \cos C = \frac{1}{2}\frac{2\sin(A-B)\cos(A+B)}{\sin(A-B)} = \cos(A+B) + \cos C = \cos(180-C) + \cos C$

$= -\cos C + \cos C = 0$ □

Problem 1.2. Show that in any triangle ABC $\sin\frac{A}{2} \leq \frac{a}{2\sqrt{bc}}$

Solution. $0 < A < 180, 0 \leq \frac{A}{2} < 90, \sin\frac{A}{2} > 0$
$\sqrt{\frac{1-\cos A}{2}}$ which means $\sin^2\frac{A}{2} = \frac{1-}{2}$ and from cos law,$\cos A = \frac{b^2+c^2-a^2}{2bc}$
$\sin^2\frac{A}{2} = \frac{1-\frac{b^2+c^2-a^2}{2bc}}{2} = \frac{a^2-(b-c)^2}{4bc} \leq \frac{a^2}{4bc}$ Hence, $\sin\frac{A}{2} \leq \frac{a}{2\sqrt{bc}}$ □

Problem 1.3. If $\tan A = \frac{1}{2m+1}$ and $\tan B = \frac{m}{m+1}$ where A and B then $A+B$ is equal to

Solution. note that $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B} = \frac{\frac{1}{2m+1} + \frac{m}{m+1}}{1 - (\frac{1}{2m+1})(\frac{m}{m+1})} = \frac{2m^2+2m+1}{2m^2+3m+1-m} = 1$
$\tan(A+B) = 1, A+B = \frac{pi}{4}$ □

Problem 1.4. If a is an acute angle and x is $(+)$ and $2x\sin^2\frac{a}{2} + 1 = x$ then, $\tan a =$ (with respect to x)

Solution. $2x\sin^2\frac{a}{2} + 1 = x$
$\sin^2\frac{a}{2} = \frac{x-1}{2x}$ which means $\frac{1-\cos a}{2} = \frac{x-1}{2x}$
$\cos a = \frac{1}{x}$ Hence, $\sin a = \sqrt{1-\cos^2 a} = \sqrt{1 - \frac{1}{x^2}} = \frac{\sqrt{x^2-1}}{x}$ Now you have $\sin a$ and $\cos a$
Therefore, $\tan a = \sqrt{x^2-1}$ □

Problem 1.5. Let $f : R- > R$ be defined as $f(x) = 2x + |x|$, then $f(2x) + f(-x) - f(x) =$

Solution. $f(2x) = 2(2x) + |2x| = 4x + 2|x|$
$f(-x) = -2x + |-x| = -2x + |x|$
$f(x) = 2x + |x|$ using those to get the answer
$f(2x) + f(-x) - f(x) = 4x + 2|x| + |x| - 2x - 2x - |x| = 2|x|$ □

Problem 1.6. If $f(x) = \frac{x}{\sqrt{1+x^2}}$,then $(fofof)(x) =$

Solution. note that $(fofof)(x) = (fof)(f(x)) = (fof)(\frac{x}{\sqrt{1+x^2}}) = f(f(\frac{x}{\sqrt{1+x^2}})) = f(\frac{\frac{x}{\sqrt{1+x^2}}}{\sqrt{1+\frac{x^2}{1+x^2}}})$

$f(\frac{x\sqrt{1+x^2}}{\sqrt{1+x^2}}\frac{1}{\sqrt{1+2x^2}}) = f(\frac{x}{\sqrt{1+2x^2}}) = \frac{\frac{x}{\sqrt{1+2x^2}}}{\sqrt{1+\frac{x^2}{1+2x^2}}} = \frac{x}{\sqrt{1+3x^2}}$ □

Problem 1.7. If $f(x) = \log(\frac{1+x}{1-x})$ and $g(x) = \frac{3x+x^3}{1+3x^2}$,then $fog(x) =$

Solution. $fog(x) = f(\frac{3x+x^3}{1+3x^2}) = \log(\frac{1+\frac{3x+x^3}{1+3x^2}}{1-\frac{3x+x^3}{1+3x^2}}) = \log(\frac{1+3x^2+3x+x^3}{1+3x^2-3x-x^3})$

$= \log(\frac{(1+x)^3}{(1-x)^3}) = 3\log(\frac{1+x}{1-x}) = 3f(x)$ □

Problem 1.8. Let x be a real number such that $\sin^{10} x + \cos^{10} x = \frac{11}{36}$. Then $\sin^{12} x + \cos^{12} x = \frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Solution. 21)Let $\sin^2 x = a$ and $\cos^2 x = b$. By the Pythagorean Identity and the given information,

$$\begin{cases} a+b=1, \\ a^5+b^5=\frac{11}{36}. \end{cases}$$

Exploiting symmetry, we let $a = \frac{1}{2} + c$ and $b = \frac{1}{2} - c$ for some constant c; then,

$$\left(\frac{1}{2}+c\right)^5 + \left(\frac{1}{2}-c\right)^5 = 5c^4 + \tfrac{5}{2}c^2 + \tfrac{1}{16} = \tfrac{11}{36}.$$

Re - arranging and simplifying, we find that

$$720c^4 + 360c^2 + 9 = 44 \implies 144c^4 + 72c^2 - 7 = 0.$$

Factoring the LHS yields $\left(12c^2-1\right)\left(12c^2+7\right)$; c^2 is positive by the Trivial Inequality, so $c = \pm\frac{\sqrt{3}}{6}$. Then $a, b = \frac{3\pm\sqrt{3}}{6}$, in no particular order, so

$$\sin^{12} x + \cos^{12} x = a^6 + b^6 = \left(\frac{3+\sqrt{3}}{6}\right)^6 + \left(\frac{3-\sqrt{3}}{6}\right)^6 = \frac{13}{54}.$$

The requested answer is $m + n = 13 + 54 = \boxed{067}$. □

Problem 1.9. In triangle ABC, point D divides side $\overline{AC}$ so that $AD : DC = 1 : 2$. Let E be the midpoint of $\overline{BD}$ and let F be the point of intersection of line BC and line AE. Given that the area of $\triangle ABC$ is 360, what is the area of $\triangle EBF$?

Solution. Using $AD : DC$ ratio, we get $[ADB] = 120$ and $[CDB] = 240$, so $[AEB] = 60$ and $[BEC] = 120$.

Letting $[BEF] = x$, we obtain $\frac{x}{60} = \frac{120-x}{180}$, or $x = \boxed{30}$. □

Problem 1.10. Find all real numbers x that satisfy the equation

$$\frac{x-2020}{1} + \frac{x-2019}{2} + \cdots + \frac{x-2000}{21} = \frac{x-1}{2020} + \frac{x-2}{2019} + \cdots + \frac{x-21}{2000},$$

Solution. Because the equation is a linear equation, that the only possible answer is 2021 □

Problem 1.11. Let $f(x) = \frac{ax+b}{cx+d}$,and $f \circ f(x) = x$
For $a = -22$,then $d =$

Solution. $\because f \circ f(x) = x$ Then, $f(x) = f^{-1}(x)$ Lets find $f^{-1}x$

$$y = \frac{ax+b}{cx+d} \to x = \frac{ay+b}{cy+d}$$

$$x(cy+d) = ay+b \to cxy+dx = ay+b \to cxy-ay = b-dx \to y = \frac{b-dx}{cx-a}$$

$$\frac{b-dx}{cx-a} = \frac{ax+b}{cx+d}$$

Here, appears clearly that $a = -d$ which yields the answer to 22 □

Problem 1.12. Find all $x, y \in \left(0, \frac{\pi}{2}\right)$ such that

$$\frac{\cos x}{\cos y} = 2\cos^2 y,$$
$$\frac{\sin x}{\sin y} = 2\sin^2 y.$$

Then, prove that $x = y$

Solution. Suming up we get:

$$\frac{\cos x}{\cos y} + \frac{\sin x}{\sin y} = 2 \Leftrightarrow \sin(x+y) = \sin 2y \Rightarrow x+y = 2y$$

$\Rightarrow x = y$ □

Problem 1.13. If it were two hours later, it would be half as long until midnight as it would be if it were an hour later than it is now. What time is it now?

Solution. $\boxed{\textbf{Answer: 9 p.m.}}$
First, we will use the 24-hour system, so that midnight is at 24. We can, then, represent the problem in an equation as follows

$$24 - (x+2) = (\frac{1}{2})(24 - (x+1))$$

The left hand side represent that time between midnight and two hours after the current time. The right hand side represents that the time between midnight and an hour after the current is half the time of the left hand side. Therefore, we can easily solve for x here and get that $x = 21$, which means it is 9 p.m. □

Problem 1.14. Given the following system of equations

$$(x+y)^2 - z^2 = 4$$

$$(y+z)^2 - x^2 = 9$$

$$(z+x)^2 - y^2 = 36$$

Then, find $x + y + z$

Solution. Add three equation together

$$x^2 + y^2 + z^2 + 2xy + 2yz + 2xz = 49$$

$$(x + y + z)^2 = 49$$

$$x + y + z = \pm 7$$

□

Problem 1.15. Evaluate

$$(\frac{2^3 - 1}{2^3 + 1} \cdot \frac{3^3 - 1}{3^3 + 1} \cdots \frac{100^3 - 1}{100^3 + 1})$$

The expression can be written as $\frac{a}{b}$ where a and b are relatively co-prime integers. Find $a + b$

Solution. Simplify

$$\prod_{n=2}^{100} \frac{n^3 - 1}{n^3 + 1} = \prod_{n=2}^{100} \frac{(n-1)(n^2 + n + 1)}{(n+1)(n^2 - n + 1)}$$

$$= (\frac{1}{3} \cdot \frac{7}{3}) \cdot (\frac{2}{4} \cdot \frac{13}{7}) \cdots (\frac{98}{100} \cdot \frac{9901}{9703}) \cdot (\frac{99}{101} \cdot \frac{10101}{9901})$$

As shown it is a telescoping were the terms are cancelled togther

$$= (\frac{1 \cdot 2}{101 \cdot 100} \cdot \frac{10101}{3}) = \frac{3367}{5050}$$

which yields the answer to 8417 □

Problem 1.16. Solve for x

$$4^x + 9^x + 49^x = 6^x + 14^x + 21^x$$

Solution. It can be written as

$$2^x \cdot 2^x + 3^x \cdot 3^x + 7^x \cdot 7^x = 2^x \cdot 3^x + 2^x \cdot 7^x + 7^x \cdot 3^x$$

Let $2^x = a$, $3^x = b$, and $7^x = c$ Then,

$$ab + bc + ac = a^2 + b^2 + c^2$$

The only solution for this equation when $a = b = c$ Which means that $2^x = 3^x = 7^x$. The only possible solution is $x = 0$ □

Problem 1.17.

$$k(x + \frac{x}{2}) = \frac{x^3 + 1}{x} + \frac{x^3 + 8}{2x^2} + 3$$

Find $k(1)$

Solution. Simplify the expression

$$k(\frac{3x}{2}) = x^2 + \frac{1}{x} + \frac{x}{2} + \frac{4}{x^2} + 3$$

Let $u = \frac{3x}{2} \to x = \frac{2x}{3}$

$$k(u) = (\frac{2u}{3})^2 + \frac{1}{\frac{2}{3}u} + \frac{\frac{2}{3}u}{2} + \frac{4}{(\frac{2}{3}u)^2} + 3$$

Then,

$$k(1) = \frac{4}{9} + \frac{3}{2} + \frac{1}{3} + 6 + 3 = \frac{203}{18}$$

□

Problem 1.18. Solve the inequality

$$\sqrt{x-1} + \sqrt{x^2-1} < \sqrt[3]{x^3}$$

Solution. We can deduce, first, that the domain is $x \in [1, \infty)$. Since both sides are nonnegative, we can square both sides and obtain that

$$2\sqrt{x^3 - x^2 - x + 1} < x^3 - x^2 - x + 2$$

Let $\sqrt{x^3 - x^2 - x + 1} = y$. So, we get that

$$(y-1)^2 > 0$$

which is equivalent to saying that

$$\sqrt{x^3 - x^2 - x + 1} \neq 1$$

To solve the latter expression, we let that $\sqrt{x^3 - x^2 - x + 1} = 1$ We get that the solutions for x are 0 and $\frac{1 \pm \sqrt{5}}{2}$. Therefore, we can deduce that the solutions for x are

$$x \in \left[1, \frac{1 \pm \sqrt{5}}{2}\right) \bigcup \left(\frac{1 \pm \sqrt{5}}{2}, \infty\right)$$

□

Problem 1.19. Evaluate the following expression

$$\left(\frac{x\sqrt{x}}{\sqrt{1-x^3}} + \frac{\sqrt{1-x^3}}{x\sqrt{x}}\right)^{-1}$$

if

$$x = \sqrt[3]{\frac{a - \sqrt{a^2 - b^2}}{2a}}$$

Where $a, b \in R$ are such that $0 <\mid a \mid \geq \mid b \mid > 0$

Solution. $\boxed{\textbf{Answer: } \dfrac{\mid b \mid}{2 \mid a \mid}}$

First, we can simplify the fraction within the expression. It is implified to

$$\left(\frac{1}{\sqrt{x^3(1-x^3)}}\right)^{-1}$$

which is equal to

$$\sqrt{x^3(1-x^3)}$$

Substituting with the value of x and simplifying the fractions, we get that it is equal to

$$\sqrt{\frac{a^2-a^2+b^2}{4a^2}}$$

Therefore, the final answer is

$$\frac{\mid b \mid}{2 \mid a \mid}$$

□

Problem 1.20.

$$a+\frac{1}{b+\frac{1}{c+\frac{1}{d}}}=\frac{29}{12}$$

Then, $a^3+b^3+c^3+d^3$

Solution.

$$\frac{29}{12}=2+\frac{5}{12}$$
$$=2+\frac{1}{\frac{12}{5}}=2+\frac{1}{2+\frac{2}{5}}$$
$$=2+\frac{1}{2+\frac{1}{\frac{5}{2}}}=2+\frac{1}{2+\frac{1}{2+\frac{1}{2}}}$$

Which means that $a=b=c=d=2$. Then,

$$4\cdot 2^3=32$$

□
